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Generalised chiral null models and rotating string backgrounds

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Abstract

We consider an extension of a special class of conformal sigma models ('chiral null models') which describe extreme supersymmetric string solutions. The new models contain both 'left' and 'right' vector couplings and should correspond to non-BPS backgrounds. In particular, we discuss a conformal six-dimensional model which is a combination of fundamental string and 5-brane models with the two extra couplings representing rotations in the orthogonal planes. If the two rotation parameters are independent the resulting background is found to be either singular or not asymptotically flat. The non asymptotically flat solution has a regular short distance limit described by a 'twisted' product of $SL(2,R)$ and $SU(2)$ WZW theories with two twist parameters mixing the isometric Euler angles of $SU(2)$ with a null direction of $SL(2,R)$.

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1 Introduction

Superstring solutions with vanishing R-R background fields are described by conformal σ-models [1, 2]. Related solutions with non-vanishing R-R fields can be obtained, e.g., by applying $SL(2,Z)$ duality of $D = 10$ type IIB superstring theory [3]. Knowing a σ-model description of a given NS-NS string solution is important since many of its features can be (at least, qualitatively) understood using conformal field theory considerations (see, e.g., [4, 5, 6]). This may complement the picture obtained for the corresponding R-R solution using D-brane approach (see, e.g, [7, 8, 9, 10]).

An important class of exact (supersymmetric, BPS saturated) string solutions is described by a special σ-model – 'chiral null model' (CNM) [11]. Here the equations on background fields obtained by imposing the conformal invariance condition are decoupled Laplace-type equations. The solutions are expressed in terms of harmonic functions, i.e. can be freely superposed (in agreement with the BPS nature of resulting field configurations). Remarkably, the 'pure-electric' CNM with flat transverse space [11] admits a 'dyonic' generalisation [12, 5, 6] where the transverse space is represented by a non-trivial $N = 4$ supersymmetric σ-model .

CNM is characterised by the presence of only certain special chiral couplings in the action. It is of interest to study what happens when one introduces extra integrable marginal perturbations getting as a result a mixture of 'left' and 'right' couplings (which, in general, may break supersymmetry). The corresponding conformal invariance conditions will no longer factorize into a simple set of Laplace equations, i.e. one will no longer have a simple BPS-type superposition principle. These models may describe certain 'non-BPS' solutions which may still have some special properties (e.g. may still be exact to all orders). An example of such model will be discussed below. It presumably still corresponds to extremal field configurations but with less (than in standard CNM case) or no supersymmetry.

Our investigation of such models was partly motivated by an attempt to construct 'rotational' generalisations of $D = 6$ CNM's describing $D = 4$ [12, 5] and $D = 5$ [6] extreme dyonic black holes (thus providing exact string-theory generalisations of the solutions of leading-order effective equations in [13] and [7]). These $D = 6$ conformal models have a short-distance (horizon or 'throat' [14, 15]) $r \to 0$ region described by (a factor of) $SL(2,R) \times SU(2)$ WZW theory. The latter is remarkable in that in the supersymmetric case it has the free value $c_{eff} = 6$ of the central charge and thus the dilaton is constant at the 'throat'.

Adding one rotational parameter to a $D = 5$ solution parametrised by 3 independent charges (two electric and one 'magnetic') can be understood [6] as adding a perturbation ($\partial u \bar{J}_3$) which 'mixes' the Cartan $\bar{J}_3$ curent of $SU(2)$ with an isometric (Gauss decomposition) coordinate u of $SL(2,R)$. It can be induced by shifting one of the isometric Euler angles of $SU(2)$, $\psi \to \psi + q_1 u$.[2] The 'integrated' CNM which extrapolates such perturbed throat region model to finite r describes the 4-parameter

[2]Closely related 'magnetic' models were discussed in [16, 17, 18].

(3 charges and two equal angular momenta in two orthogonal planes) extreme $D = 5$ black hole solution [6] which generalises the solution of the leading-order effective string equations originally found in [10] (where the 3 non-trivial charge parameters were all equal).

A natural idea is to generalise the throat region model further shifting also the *second* Euler angle of $SU(2)$, $\varphi \to \varphi + q_2 u$, thus inducing the perturbation $\bar\partial u J_3$ which should correspond to the second independent rotation parameter in the resulting $D = 5$ background. For conformal invariance one needs also to add non-trivial $O(q_1 q_2 \partial u \bar\partial u)$ term. Similar string model was considered in [18] in the context of continuous supersymmetry breaking by 'magnetic' backgrounds.

The σ-model which extends such (q_1, q_2)-perturbed $SL(2,R) \times SU(2)$ WZW model to finite r is a simple generalisation of CNM discussed below. We shall find that for $q_2 \neq 0$ the conformal invariance conditions do not have solutions which describe asymptotically flat at $r \to \infty$ *and* non-singular at $r \to 0$ backgrounds. This implies that there are no non-singular extremal $D = 5$ black holes with two independent rotational parameters (in agreement with similar remarks in [10]).

The solution that reduces to the regular (q_1, q_2)-perturbed throat region model is not asymptotically flat and describes a rotating magnetic universe. Analogous solutions were discussed in [11, 19].[3] Given that the background with $q_1 = q_2 = 0$ has a simple 'fundamental string + 5-brane' interpretation it may be of interest to study the D-brane description of the direct analog of this $q_1 q_2 \neq 0$ solution which has non-vanishing R-R fields. The D-brane picture in [10] based on two $U(1)$ currents of $SU(2) \times SU(2)$ KM algebra of underlying 2d conformal model has a striking similarity with the above description of the throat region model with the two ('left' and 'right') Cartan current perturbations.

2 Review

The 'standard' CNM with curved transverse part is defined by the Lagrangian [11][4]

$$L = F(x)\partial u \left[\bar\partial v + K(x)\bar\partial u + 2\mathcal{A}_i(x)\bar\partial x^i\right] + \tfrac{1}{2}\mathcal{R}\ln F(x) + L_\perp \ , \tag{1}$$

$$L_\perp = (G_{ij} + B_{ij})(x)\partial x^i \bar\partial x^j + \mathcal{R}\phi(x) \ . \tag{2}$$

There exists a renormalisation scheme in which (1) is conformal to all orders in α' provided the 'transverse' σ-model (2) is conformal and the functions $F^{-1}, K, \mathcal{A}_i, \Phi$ satisfy certain conformal invariance conditions. The simplest tractable case is when

[3] There is also a similarity to Kaluza-Klein Melvin model [20, 16] which can be obtained by shifting an angular coordinate by a Kaluza-Klein one.

[4] We shall ignore possible u-dependence of couplings. When both K and $\mathcal{A}_i$ depend on u one can redefine v to absorb K into $\mathcal{A}_i$ [11].

the transverse theory is either flat or has at least $(4,0)$ extended world-sheet supersymmetry so that the conformal invariance conditions essentially preserve their 1-loop form, i.e. are the 'Laplace' equations in the 'transverse' background [11, 5]

$$\partial_i(e^{-2\phi}\sqrt{G}G^{ij}\partial_j F^{-1}) = 0 \ , \qquad \partial_i(e^{-2\phi}\sqrt{G}G^{ij}\partial_j K) = 0 \ , \tag{3}$$

$$\hat{\nabla}_{+i}(e^{-2\phi}\mathcal{F}^{ij}) = 0 \ , \quad i.e. \quad \partial_i(e^{-2\phi}\sqrt{G}\mathcal{F}^{ij}) - \tfrac{1}{2}e^{-2\phi}\sqrt{G}H^{kij}\mathcal{F}_{ki} = 0 \ , \tag{4}$$

where $\hat{\Gamma}^i_{jk} = \Gamma^i_{jk} + \frac{1}{2}H^i{}_{jk}$, $\mathcal{F}_{ij} \equiv \partial_i\mathcal{A}_j - \partial_j\mathcal{A}_i$, $H_{ijk} = 3\partial_{[i}B_{jk]}$. For example, in the case when the transverse space is described by the '5-brane' model [14] $(i,j,... = 1,2,3,4)$

$$L_\perp = f(x)\partial x^i\bar{\partial}x^i + B_{ij}(x)\partial x^i\bar{\partial}x^j + \mathcal{R}\phi(x) \ , \tag{5}$$

$$G_{ij} = f(x)\delta_{ij} \ , \qquad H^{ijk} = -\frac{2}{\sqrt{G}}\epsilon^{ijkp}\partial_p\phi \ , \qquad \phi = \tfrac{1}{2}\ln f \ , \quad \partial^i\partial_i f = 0 \ ,$$

the equations for F, K, f become the free 4-d Laplace equations $\partial^i\partial_i F^{-1} = 0$, $\partial^i\partial_i K = 0$, while the equation for $\mathcal{A}_i$ can be re-written as follows [6]

$$\partial_i(e^{-2\phi}\sqrt{G}\mathcal{F}^{ij}_+) = 0 \ , \quad i.e. \quad \partial_i(f^{-1}\mathcal{F}^{ij}_+) = 0 \ , \tag{6}$$

$$\mathcal{F}^{ij}_+ \equiv \mathcal{F}^{ij} + \mathcal{F}^{*ij} \ , \qquad \mathcal{F}^{*ij} = \frac{1}{2\sqrt{G}}\epsilon^{ijkl}\mathcal{F}_{kl} \ .$$

This equation can be solved, e.g., by imposing $\mathcal{F}^{ij}_+ = 0$ which is again a linear flat-space equation (the scale factor of conformally flat G_{ij} drops out) and should describe supersymmetric BPS-type backgrounds. A particular solution is the 4-parameter rotating dyonic $D = 5$ black hole [6] which generalises the solution of [10].

In general the above equation for $\mathcal{A}_i$ depends on the harmonic function f of the transverse theory and does not reduce to a flat-space Maxwell equation. This is illustrated by a related example with an extra 'left' electric charge added to a dyonic model with two electric and two magnetic functions which was discussed in [5]. The resulting equation for the new electric charge function $A(x)$ (a component of $\mathcal{A}_i$) was not a free flat-space Laplace equation. Generically there is an interaction between electric (longitudinal or time-like) and magnetic (transverse or space-like) parts of the model which cannot be switched off.[5]

In addition to the 'longitudinal'–'transverse' coupling one can also introduce an interaction between 'left-moving' and 'right-moving' coupling functions. The simplest example is provided by the 'plane-wave' model [21]

$$L = \partial u\bar{\partial}v + K(x)\partial u\bar{\partial}u + 2\mathcal{A}_i(x)\partial u\bar{\partial}x^i + 2\bar{\mathcal{A}}_i(x)\partial x^i\bar{\partial}u + L_\perp \ , \tag{7}$$

[5]This raises the following question: given a type II or heterotic theory compactified to $D = 4$, are there 'dyonic' solutions parametrised by more than two general (not necessarily one-center) harmonic functions of 3 spatial coordinates?

or its u-dual [11] ($\tilde{F} = K^{-1},\ \tilde{\mathcal{A}}_i = \bar{\mathcal{A}}_i,\ v' = v + \tilde{u}$)

$$L = \tilde{F}(x)[\partial\tilde{u} - 2\tilde{\mathcal{A}}_i(x)\partial x^i][\bar{\partial}v' + 2\mathcal{A}_j(x)\bar{\partial}x^j] + \tfrac{1}{2}\mathcal{R}\ln\tilde{F}(x) + L_\perp \ . \tag{8}$$

which are conformal provided

$$\partial_i[e^{-2\phi}\sqrt{G}\mathcal{F}^{ij}(\mathcal{A})] - \tfrac{1}{2}e^{-2\phi}\sqrt{G}H^{kij}\mathcal{F}_{ki}(\mathcal{A}) = 0\ , \tag{9}$$

$$\partial_i[e^{-2\phi}\sqrt{G}\mathcal{F}^{ij}(\bar{\mathcal{A}})] + \tfrac{1}{2}e^{-2\phi}\sqrt{G}H^{kij}\mathcal{F}_{ki}(\bar{\mathcal{A}}) = 0\ ,$$

$$\partial_i(e^{-2\phi}\sqrt{G}G^{ij}\partial_j K) - 2e^{-2\phi}\sqrt{G}\mathcal{F}^{ij}(\mathcal{A})\mathcal{F}_{ij}(\bar{\mathcal{A}}) = 0\ .$$

The equation for K (or $\tilde{F}^{-1}$) is no longer a free Laplace equation, i.e. there is no simple superposition principle. In contrast to the standard plane-wave case [22] one may also expect higher α'-corrections (unless there exists a special 'exact' scheme). The special cases in which the simplicity is restored correspond to taking the two vector field strengths to be constant or choosing an anti-self-dual $\mathcal{F}_{ij}(\mathcal{A})$ and self-dual $\mathcal{F}_{ij}(\bar{\mathcal{A}})$ so that $\mathcal{F}^{ij}(\mathcal{A})\mathcal{F}_{ij}(\bar{\mathcal{A}}) = 0$, i.e. the equation for K becomes again the homogeneous Laplace one.

3 Generalised CNM

Below we shall consider the following generalisation of the above models (1),(7)

$$L = F(x)[\partial u\bar{\partial}v + K(x)\partial u\bar{\partial}u + 2\mathcal{A}_i(x)\partial u\bar{\partial}x^i + 2\bar{\mathcal{A}}_i(x)\partial x^i\bar{\partial}u] + \tfrac{1}{2}\mathcal{R}\ln F(x) + L_\perp\ . \tag{10}$$

The u-dual of this model is a generalisation of (8)

$$\tilde{L} = \tilde{F}(x)[\partial\tilde{u} - 2\tilde{\mathcal{A}}_i(x)\partial x^i][\bar{\partial}v + \tilde{K}(x)\bar{\partial}\tilde{u} + 2\mathcal{A}_j(x)\bar{\partial}x^j] + \tfrac{1}{2}\mathcal{R}\ln\tilde{F}(x) + L_\perp\ , \tag{11}$$

$$\tilde{F} = K^{-1}, \quad \tilde{K} = F^{-1}, \quad \tilde{\mathcal{A}}_i = F\bar{\mathcal{A}}_i\ . \tag{12}$$

In contrast to the original CNM (1) the generalised model (10) is no longer self-dual. This is also an indication of its 'non-BPS' nature.

The 1-loop conditions of conformal invariance of (10) can be derived from the standard equation $\hat{R}_{-\mu\nu} + 2\hat{\nabla}_{-\mu}\hat{\nabla}_{-\nu}\Phi = 0$ for a general σ-model ($x^\mu = (u, v, x^i)$). A more illuminating way of obtaining them is to add source terms ($\Delta L = -\partial u\bar{\partial}V - \bar{\partial}v\partial U$) and to integrate out u and v as in [11].[6] As a result, we get

$$L' = K\partial U\bar{\partial}\partial^{-1}(F^{-1}\partial U) - F^{-1}\partial U\bar{\partial}V \tag{13}$$

$$+\ 2\mathcal{A}_i\bar{\partial}x^i\partial U + 2F\bar{\mathcal{A}}_i\partial x^i\bar{\partial}\partial^{-1}(F^{-1}\partial U) + L_\perp\ .$$

[6]The fact that the action is still linear in v so that the null coordinates u and v can be integrated out strongly indicates that there still exist a scheme in which the 1-loop conformal invariance conditions are exact to all orders (provided this is also true for the transverse part of the model).

The $O(\ln F)$ dilaton term cancels out. Since U and V are external fields it is then straightforward to study conditions of conformal invariance with respect to the transverse coordinates only. One finds the following set of equations which generalises (3),(4),(9)

$$\partial_i(\sqrt{G}e^{-2\phi}G^{ij}\partial_j F^{-1}) = 0 \ , \tag{14}$$

$$\partial_i(\sqrt{G}e^{-2\phi}G^{ij}\partial_j K) - 2\sqrt{G}e^{-2\phi}\mathcal{F}^{ij}(\tilde{\mathcal{A}})[\mathcal{F}_{ij}(C) - F^{-1}\mathcal{F}_{ij}(\tilde{\mathcal{A}})] = 0 \ , \tag{15}$$

$$\partial_i[\sqrt{G}e^{-2\phi}\mathcal{F}^{ij}(\tilde{\mathcal{A}})] + \tfrac{1}{2}\sqrt{G}e^{-2\phi}H^{kij}\mathcal{F}_{ki}(\tilde{\mathcal{A}}) = 0 \ , \tag{16}$$

$$\partial_i\{\sqrt{G}e^{-2\phi}[\mathcal{F}^{ij}(C) - 2F^{-1}\mathcal{F}^{ij}(\tilde{\mathcal{A}})]\} - \tfrac{1}{2}\sqrt{G}e^{-2\phi}H^{kij}\mathcal{F}_{ki}(C) = 0 \ . \tag{17}$$

Here the gauge-invariant field strengths correspond to the following potentials[7]

$$\tilde{\mathcal{A}}_i \equiv F\bar{\mathcal{A}}_i \ , \qquad C_i \equiv \mathcal{A}_i + \bar{\mathcal{A}}_i \ . \tag{18}$$

3.1 Flat transverse space: charged fundamental string

Let us first discuss an example of generalised CNM (10) with flat transverse space but $\bar{\mathcal{A}}_i \neq 0$ which corresponds to a generalisation of the charged $D = 5$ fundamental string solution [23]. Splitting the coordinates x^i into $N = 5$ compact toroidal ones y^n and 3 non-compact spatial ones x^s we may set

$$\mathcal{A} = \mathcal{A}_n(x)dy^n \ , \quad \bar{\mathcal{A}} = \bar{\mathcal{A}}_n(x)dy^n \ , \quad \tilde{\mathcal{A}}_n = F\bar{\mathcal{A}}_n \ , \quad C_n = \mathcal{A}_n + \bar{\mathcal{A}}_n \ , \tag{19}$$

and assume that all the functions will depend only on x^s. Then the equations (14)–(17) become

$$\partial^s\partial_s F^{-1} = 0 \ , \quad \partial^s\partial_s K - 4\partial^s\tilde{\mathcal{A}}_n(\partial_s C_n - F^{-1}\partial_s\tilde{\mathcal{A}}_n) = 0 \ , \tag{20}$$

$$\partial^s(\partial_s C_n - 2F^{-1}\partial^s\tilde{\mathcal{A}}_n) = 0 \ , \quad \partial^s\partial_s\tilde{\mathcal{A}}_n = 0 \ . \tag{21}$$

Though for $\bar{\mathcal{A}}_n \neq 0$ these equations no longer reduce to free Laplace equations, their general solution can still be expressed in terms of $2N + 2 = 12$ independent harmonic functions.[8] Choosing all harmonic functions to be 1-center ones we find

$$F^{-1} = 1 + Qr^{-1}, \quad \tilde{\mathcal{A}}_n = p_n r^{-1}, \quad C_n = (p_n + q_n)r^{-1} + p_n Q r^{-2}, \tag{22}$$

$$\mathcal{A}_n = q_n r^{-1} \ , \qquad \bar{\mathcal{A}}_n = p_n r^{-1} + p_n Q r^{-2} \ ,$$
$$K = 1 + \tilde{Q}r^{-1} + 2q_n p_n r^{-2} + \tfrac{2}{3}p_n p_n Q r^{-3} \ , \quad r^2 = x^s x^s \ . \tag{23}$$

[7]As it is clear from (11) the dual action is invariant under $\tilde{u} \to \tilde{u} + h(x)$, $v \to v + s(x)$, $\mathcal{A}_m \to \mathcal{A}_m - \partial_m s - \tilde{K}\partial_m h$, $\tilde{\mathcal{A}}_m \to \tilde{\mathcal{A}}_m + \partial_m h$, so that the gauge-invariant vector field strengths are $d\tilde{\mathcal{A}}$ and $d\mathcal{A} + dF^{-1} \wedge \tilde{\mathcal{A}} = dC - F^{-1}d\tilde{\mathcal{A}}$ or just $d\tilde{\mathcal{A}}$ and dC.

[8]In the heterotic string case one may introduce extra 16 'right' vector couplings leading to extra 16 harmonic functions.

When both q_n and p_n are non-vanishing the off-diagonal components of the $D=10$ metric and the antisymmetric tensor are different, i.e. one finds the fundamental string background with both 'left' and 'right' charges. Separating the 'internal' y^n-part $((\partial y_n + FC_n\partial u)^2 + ...)$ the rest of the action becomes the neutral fundamental string CNM with redefined function $K \to K' = K - FC_n^2$. The reduction to $D=4$ then leads to a family of electrically charged black holes parametrised by the charges $Q, \tilde{Q}, p_n, q_n$ (12 in type II or 12 +16 in the heterotic string case). The corresponding $D=4$ Einstein-frame metric is given by

$$ds_E^2 = -\lambda(r)dt^2 + \lambda^{-1}(r)(dr^2 + r^2 d\Omega) \ , \tag{24}$$

$$\lambda^{-2} = F^{-1}K - C_n^2 = 1 + (Q+\tilde{Q})r^{-1} + (Q\tilde{Q} - q_n^2 - p_n^2)r^{-2} - \tfrac{4}{3}p_n^2 Q r^{-3} - \tfrac{1}{3}p^2Q^2r^{-4} \ .$$

For $p_n \neq 0$ the background is more singular than in the usual $p_n = 0$ case. This seems to be a general pattern: solutions with non-vanishing $\bar{A}_i$ coupling have stronger short-distance singularities (see also below).

3.2 5-brane model as transverse theory

In what follows we shall consider an example with a curved transverse theory represented by the 5-brane model (5). Since here $\sqrt{G}e^{-2\phi}H^{kij} = \epsilon^{kijl}\partial_l e^{-2\phi}$ one can put the equations (14)–(17) into the form (repeated indices are now contracted using flat space metric)

$$\partial^2 F^{-1} = 0 \ , \qquad \partial^2 K - 2f^{-1}\mathcal{F}_{ij}(\tilde{\mathcal{A}})[\mathcal{F}_{ij}(C) - F^{-1}\mathcal{F}_{ij}(\tilde{\mathcal{A}})] = 0 \ , \tag{25}$$

$$\partial_i(f^{-1}[\mathcal{F}_{ij}(\tilde{\mathcal{A}}) - \mathcal{F}^*_{ij}(\tilde{\mathcal{A}})]) = 0 \ , \tag{26}$$

$$\partial_i(f^{-1}[\mathcal{F}_{ij}(C) + \mathcal{F}^*_{ij}(C) - 2F^{-1}\mathcal{F}_{ij}(\tilde{\mathcal{A}})]) = 0 \ . \tag{27}$$

Note the last two equations have the following special solution

$$\mathcal{F}_{ij}(\tilde{\mathcal{A}}) = \mathcal{F}^*_{ij}(\tilde{\mathcal{A}}) \ , \qquad \mathcal{F}_{ij}(\tilde{\mathcal{A}}) = \tfrac{1}{2}F[\mathcal{F}_{ij}(C) + \mathcal{F}^*_{ij}(C)] \ , \tag{28}$$

for which the equation for K becomes again the free Laplace equation

$$\partial^2 K = f^{-1}\mathcal{F}_{ij}(\tilde{\mathcal{A}})[\mathcal{F}_{ij}(C) - \mathcal{F}^*_{ij}(C)] = 0 \ . \tag{29}$$

For $F=1$ this solution reduces to the case of self-dual $\bar{\mathcal{A}}_i$ and anti-self-dual $\mathcal{A}_i$ mentioned at the end of Section 2.

4 $D=6$ conformal model with 'rotations' in transverse planes

Let us now solve the equations (25)–(27) using the $SO(4)$ symmetric choice for the harmonic functions f and F^{-1}

$$f = 1 + Pr^{-2} \ , \qquad F^{-1} = 1 + Qr^{-2} \ , \qquad r^2 = x^i x^i \ , \tag{30}$$

and the following ansatz for $\mathcal{A}_i, \bar{\mathcal{A}}_i$ with rotational symmetry in the two orthogonal planes

$$\mathcal{A}_i dx^i = h_1(r)\sin^2\theta d\varphi + h_2(r)\cos^2\theta d\psi \ , \quad \bar{\mathcal{A}}_i dx^i = \bar{h}_1(r)\sin^2\theta d\varphi + \bar{h}_2(r)\cos^2\theta d\psi \ . \quad (31)$$

Here the four spatial transverse coordinates x^i are chosen as $x^1 + ix^2 = r\sin\theta e^{i\varphi}$, $x^3 + ix^4 = r\cos\theta e^{i\psi}$, so that the 5-brane model Lagrangian $L_\perp$ in (5) takes the form

$$L_\perp = f(r)[\partial r \bar{\partial} r + r^2(\partial\theta\bar{\partial}\theta + \sin^2\theta\partial\varphi\bar{\partial}\varphi + \cos^2\theta\partial\psi\bar{\partial}\psi)] \quad (32)$$

$$+ \ \tfrac{1}{2}P\cos2\theta(\partial\varphi\bar{\partial}\psi - \bar{\partial}\varphi\partial\psi) + \tfrac{1}{2}\mathcal{R}\ln f(r) \ .$$

It then follows from (25) that in general K should depend also on θ

$$K(r,\theta) = K_1(r) + K_2(r)\cos2\theta \ . \quad (33)$$

For the gauge potential of the form in (31) one finds the following non-vanishing components of the field strengths (prime denotes the derivative over r and $\mathcal{F}^{ij}_\pm(\mathcal{A}) \equiv \mathcal{F}^{ij}(\mathcal{A}) \pm \mathcal{F}^{*ij}(\mathcal{A})$)

$$\mathcal{F}^{r\varphi}_\pm = r^{-3}(rh_1' \pm 2h_2), \qquad \mathcal{F}^{r\psi}_\pm = r^{-3}(rh_2' \pm 2h_1), \quad (34)$$

$$\mathcal{F}^{\theta\varphi}_\pm = r^{-4}\cot\theta(\pm rh_2' + 2h_1), \qquad \mathcal{F}^{\theta\psi}_\pm = -r^{-4}\tan\theta(\pm rh_1' + 2h_2).$$

Let us first consider the special case of $\bar{\mathcal{A}}_i = 0$ discussed in [6]. Then the general solution of the equation (27) for C_i (i.e. for $\mathcal{A}_i$) is found to be ($k_a, n_a = \mathrm{const}$)

$$h_{+1} \equiv h_1 + \bar{h}_1 = \tfrac{1}{2}(k_1+k_2)r^2 + k_1 P + \tfrac{1}{2}(n_1+n_2)r^{-2} + \tfrac{1}{3}n_2 P r^{-4} \ , \quad (35)$$

$$h_{+2} \equiv h_2 + \bar{h}_2 = \tfrac{1}{2}(k_1-k_2)r^2 + k_1 P + \tfrac{1}{2}(n_1-n_2)r^{-2} - \tfrac{1}{3}n_2 P r^{-4} \ .$$

The resulting σ-model (10) is regular at $r \to 0$ only if there is no r^{-4} pole, i.e. if $n_2 = 0$. Moreover, the background is asymptotically flat ($r \to \infty$) if $k_1 = k_2 = 0$. The resulting special solution

$$\mathcal{A}_i dx^i = n_1 r^{-2}(\sin^2\theta d\varphi + \cos^2\theta d\psi) \ , \qquad \bar{\mathcal{A}}_i = 0 \ , \quad (36)$$

describes [6] the $D=5$ rotating black hole [10] with equal angular momenta in the two orthogonal planes $J_\varphi = J_\psi = J = n_1\pi/4G_N$.

The general solution of the equation (26) for $\tilde{\mathcal{A}}_i$ is similar to (35)

$$\tilde{h}_1 \equiv F\bar{h}_1 = \tfrac{1}{2}(q_1+q_2)r^2 + q_2 P + \tfrac{1}{2}(p_1+p_2)r^{-2} + \tfrac{1}{3}p_1 P r^{-4} \ , \quad (37)$$

$$\tilde{h}_2 \equiv F\bar{h}_2 = \tfrac{1}{2}(q_1-q_2)r^2 - q_2 P + \tfrac{1}{2}(p_1-p_2)r^{-2} + \tfrac{1}{3}p_1 P r^{-4} \ .$$

The self-dual solution (28) corresponds to the case of $p_1 = 0, \ q_2 = 0$. It is easy to see that (independently of the form of the associated solutions for $\mathcal{A}_i$ and K) the

σ-model (10) has singular short distance ($r \to 0$) region unless $p_1 = p_2 = 0$. The resulting background is asymptotically flat only if $q_1 = q_2 = 0$.

We conclude that there are no *regular* asymptotically flat solutions with $\bar{\mathcal{A}}_i \neq 0$. In particular, there does not exist a non-singular 'extremal' generalisation of the special solution (36) to the case of two independent rotational parameters.

There is still a non asymptotically flat solution with a regular $r \to 0$ region which is a natural 2-parameter generalisation of the horizon region [6] of the solution (36). Relaxing the condition of asymptotic flatness, i.e. choosing

$$\tilde{\mathcal{A}}_i dx^i = [\tfrac{1}{2}(q_1+q_2)r^2 + q_2 P]\sin^2\theta d\varphi + [\tfrac{1}{2}(q_1-q_2)r^2 - q_2 P]\cos^2\theta d\psi \ , \qquad (38)$$

we find that (27) has the following general solution which is regular at $r=0$ (cf.(35))

$$h_{+1} = \tfrac{1}{2}(k_1+k_2)r^2 + k_1 P + \tfrac{1}{2}n_1 r^{-2} + fF^{-1}(q_1+q_2)r^2 \ , \qquad (39)$$

$$h_{+2} = \tfrac{1}{2}(k_1-k_2)r^2 + k_1 P + \tfrac{1}{2}n_1 r^{-2} + fF^{-1}(q_1-q_2)r^2 \ .$$

Then $\mathcal{A}_i$ is given by (31) with

$$h_1 = \tfrac{1}{2}(k_1'+k_2')r^2 + (k_1'P + q_1 Q + q_2 Q) + \tfrac{1}{2}n_1' r^{-2} \ , \qquad (40)$$

$$h_2 = \tfrac{1}{2}(k_1'-k_2')r^2 + (k_1'P + q_1 Q - q_2 Q) + \tfrac{1}{2}n_1' r^{-2} \ ,$$

$$k_1' = k_1 + q_1, \quad k_2' = k_2 + q_2, \quad n_1' = n_1 + 2QPq_1 \ .$$

The function K (33) can be determined from (25), i.e. from

$$r^2(r^3 K_1')' = 2f^{-1}[r^2\tilde{h}_1' h_{+1}' + r^2\tilde{h}_2' h_{+2}' + 8\tilde{h}_1 h_{+1} + 8\tilde{h}_2 h_{+2} \qquad (41)$$

$$- F^{-1}(r^2\tilde{h}_1'\tilde{h}_1' + r^2\tilde{h}_2'\tilde{h}_2' + 8\tilde{h}_1\tilde{h}_1 + 8\tilde{h}_2\tilde{h}_2)] \ ,$$

$$(r^3 K_2')' - 8rK_2 = r^4[r^{-5}(r^4 K_2)']' = -2rf^{-1}(\tilde{h}_1' h_{+1}' - \tilde{h}_2' h_{+2}') \ . \qquad (42)$$

A particular solution corresponding to the special case of $k_1 = k_2 = n_1 = 0$ is

$$K = c_0 + \tilde{Q} r^{-2} + F^{-1} f r^2 (q_1^2 + q_2^2 - 2q_1 q_2 \cos 2\theta) \ . \qquad (43)$$

5 Short-distance region: relation to $SL(2,R) \times SU(2)$ WZW theory

The $r \to 0$ limit of the $D=6$ conformal model discussed in the previous section is described by the following Lagrangian

$$L = e^{-z}\partial u \bar{\partial} v + [Q^{-1}\tilde{Q} - P(q_1^2 + q_2^2 - 2q_1 q_2 \cos 2\theta)]\partial u \bar{\partial} u \qquad (44)$$

$$+ 2Pq_1 \partial u(\sin^2\theta\bar{\partial}\varphi + \cos^2\theta\bar{\partial}\psi) + 2Pq_2\bar{\partial}u(\sin^2\theta\partial\varphi - \cos^2\theta\partial\psi)$$

$$+ P[\tfrac{1}{4}\partial z\bar\partial z + \partial\theta\bar\partial\theta + \sin^2\theta\partial\varphi\bar\partial\varphi + \cos^2\theta\partial\psi\bar\partial\psi$$
$$+ \tfrac{1}{2}\cos 2\theta(\partial\varphi\bar\partial\psi - \bar\partial\varphi\partial\psi)] \ , \qquad z \equiv -\ln(Q^{-1}r^2) \ .$$

The total dilaton $\Phi = \frac{1}{2}\ln(Ff)$ is constant in the $r \to 0$ limit. This model is related to the $SL(2,R) \times SU(2)$ WZW model[9]

$$L = e^{-z}\partial u\bar\partial v + Q^{-1}\tilde Q\partial u\bar\partial u + \tfrac{1}{4}P\partial z\bar\partial z \tag{45}$$
$$+ P(\partial\theta\bar\partial\theta + \sin^2\theta\partial\varphi\bar\partial\varphi + \cos^2\theta\partial\psi\bar\partial\psi) + \tfrac{1}{2}\cos 2\theta(\partial\varphi\bar\partial\psi - \bar\partial\varphi\partial\psi)] \ ,$$

by the formal coordinate shift[10]

$$\varphi \to \varphi + (q_1 + q_2)u \ , \qquad \psi \to \psi + (q_1 - q_2)u \ . \tag{46}$$

Thus the throat region model (44) is locally a direct product (but is globally non-trivial for generic values of q_1, q_2). It has broken supersymmetry [18] unless the twists q_1, q_2 take quantised values (for which the resulting string model becomes equivalent to the model with $q_1 = q_2 = 0$). Note that the positivity of the coefficient of $\partial u\bar\partial u$ term in both (44) and (45) implies a bound on $|q_1 - q_2|$: $(q_1 - q_2)^2 < Q^{-1}P^{-1}\tilde Q$.

Though this model has two natural 'twist' parameters, we have found above that for $q_2 \neq 0$ it does not have an extension to finite r which is asymptotically flat. It may be of interest to study related non asymptotically flat models from the point of view of their possible 'magnetic' interpretation, given that the corresponding backgrounds can be transformed into solutions with non-vanishing R-R fields which have straightforward D-brane interpretation.

Non-extremal 5-dimensional black hole solutions with two rotational parameters were recently constructed in [24, 25].

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[9]The standard $SL(2,R)$ WZW Lagrangian written in the Gauss decomposition parametrisation can be put in the 'non-standard' form which appears here by the following field redefinition [11]

$$e^{-2x'}\partial u'\bar\partial v' + \partial x'\bar\partial x' = e^{-2x}\partial u\bar\partial v + \partial u\bar\partial u + \partial x\bar\partial x, \quad u' = \tfrac{1}{2}e^{2u}, \quad v' = v - e^{2u}, \quad x' = x + u.$$

The levels of $SU(2)$ and $SL(2,R)$ WZW models are both equal to P/α'.

[10]The standard $SU(2)$ Euler angles $(\theta', \varphi', \psi')$ are related to the angular coordinates used above by

$$\theta = \tfrac{1}{2}\theta', \quad \varphi = \tfrac{1}{2}(\varphi' + \psi'), \quad \psi = \tfrac{1}{2}(\psi' - \varphi'), \quad 0 \le \theta' \le \pi, \ 0 \le \varphi' \le 2\pi, \ 0 \le \psi' \le 4\pi.$$

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